

# On the Optimal Number of Smart Dust Particles

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*Abstract*—Smart Dust particles, are small smart materials used for generating weather maps. We investigate question of the optimal number of Smart Dust particles necessary for generating precise, computationally feasible and cost effective 3-D weather maps. We also give an optimal matching algorithm for the generalized scenario, when there are  $N$  Smart Dust particles and  $M$  ground receivers.

*Keywords*—Remote sensing, smart dust, matching, optimization.

## I. INTRODUCTION

**S**MART Dust are small maple leaf like structures. On board are miniature sensors for temperature, moisture and wind profile monitoring. To relay the information they are also equipped with signal emitters. Since these leaves are very light weighted they descend slowly towards the earth's surface, and as they do, they constantly send out information about temperature, moisture and wind directions. Each leaf costs around US \$30, and is released into the atmosphere by a small auto plane [1].

The potential applications for these 'Smart Dust Particles' as pointed out in [1], are to trace wind profiles in the Bay area, and since in reality these particles fall in a 3-D environment, a possibility of constructing a 3-D weather map also exists.

A lot of research has been done towards the development of their functionality and structure which is summarized in [1], but still there exist some problems related to Smart Dust. In order to construct a 3-D weather map, we are faced with two major difficulties [2], [3].

- 1) How to map the signals from various senders to receivers, with drift or other atmospheric constraints?
- 2) What is the optimal number Smart Dust particles needed to generate a precise, computationally feasible and cost effective 3-D weather map?

We investigate both the above stated problems. We present an  $\mathcal{O}(n \cdot m)$  algorithm to optimally match  $N$  senders and  $M$  receivers in the generalized scenario, by showing the problem equivalence to maximal-Bipartite Graph Matching Problem. In [3], they made the conjecture that the number of Smart Dust particles should be  $n \sim 1/\epsilon$ , where  $\epsilon$  is the drift error. We prove this conjecture for the case of  $N$  senders and 2 receivers, by employing signal phase difference.

The direction from which the signal is received is known but not the distance, i.e. the particles can not be mapped correctly to the 3-D map, which is to be constructed from the information gathered during the particles' decent. The error in the mapping, is primarily due to the wind drifts experienced by

the particles. An asymptotically optimal matching algorithm was given in [3], which solves the matching problem, but poses another problem of determining the optimal number of Smart Dust particles that are to be released into the atmosphere. An upper bound on the number of particles to be released is important in order to generate an accurate, computationally feasible and cost effective 3-D map. We divide the paper as follows. First we will give another solution to the optimal matching of the signals between the senders and receivers, after which we will develop the mathematics necessary to obtaining the optimal number of Smart Dust particles and prove the conjecture made in [3].

## II. THE PROBLEM OF OPTIMAL MATCHING

As mentioned earlier, there exists the problem of uniquely mapping signals from various Smart Dust particles to the ground receivers. In general this matching is solvable in  $\mathcal{O}(2n \cdot \log n)$  [3]. Here, we will give another approach by transforming the problem to 'Maximal-Bipartite Graph Matching Problem' (BGMP). Our approach is slower by a factor of  $\log$ , then the solution proposed by our predecessors, but now we are able to match  $M$  receivers and  $N$  senders. Before we transform the problem to BGMP, we will quote some lemmas that are necessary for the transformation.

We can define the Bipartite Graph Matching problem as follows: A graph  $G = (V, E)$  having a set of nodes  $L$  and a set of nodes  $R$  such that  $L \cap R = \phi$ ,  $L \cup R = V$ , and  $\forall (u, v) \in E$ ,  $u \in L$  and  $v \in R$ .

*Lemma 1:* A matching of a graph  $G = (V, E)$  is a subset of edges such that no two edges are incident to the same node.

*Proof:* A matching  $M$  in a graph  $G = (V, E)$  is a subset of  $E$  such that there is no  $\{u, v_1, v_2\} \in V$  such that  $v_1 \neq v_2$  and either  $(u, v_1) \in M$  and  $(u, v_2) \in M$ , or  $(v_1, u) \in M$  and  $(v_2, u) \in M$ . ■

It turns out that using the approach of augmented paths, converges to a much simpler solution. Augmenting Path actually takes a non-maximum maximal matching and extends it by changing the pairing of some of the nodes. An augmenting path starts at an unmatched node, then alternately takes unmatched and matched edges back and forth until an unmatched node on the right is reached.

*Lemma 2:* A matching in graph  $G$  is maximum if and only if there is no augmenting path with respect to it.

*Proof:* It is clear that given a Matching  $M$ , if we have an augmented path  $P$ , then an improvement can be brought to  $M$ . This can be simply done by replacing in  $M$  the edges of  $P \cap M$  by  $P - M$ . This new  $M$  would be larger than the previous. Conversely, if  $M$  is not maximum, and there exists a larger matching  $M'$ , by considering the connected components of  $M \cup M'$ . These connected components are either alternating

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paths or alternating circuits with respect to  $M$ . At least one such component must contain more edges of  $M'$  than of  $M$ . This component is an augmenting path with respect to  $M$  [4], [5]. ■

Lemma 2 suggests the following algorithm for computing a maximum matching: start with a feasible matching  $M$ , try to find repeatedly an augmenting path  $P$ , and replace  $M$  by their Symmetric Difference Graph [6]. If there is no more path, we have found our maximal matching.

Lemma 3: Let  $G = (V, E)$  be a graph and  $|V| = n$  and  $|E| = m$ . The bipartite matching algorithm runs in worst case time  $\mathcal{O}(n \cdot m)$  for a given graph  $G = (V, E)$ .

Proof: The algorithm executes the search and augment procedures at most  $n$  times [4]. The augment procedure clearly requires  $\mathcal{O}(n)$  time. For each node  $i$ , the search procedure performs one of the following two operations at most once: it finds an even edge, or it finds an odd one. The latter operation of course requires constant time per execution. The former operation requires  $\mathcal{O}|Adj(i)|$  (where  $Adj(i)$  is the list of adjacent nodes of  $i$ ), so a total of  $\mathcal{O}(\sum_{v \in V} |Adj(i)|) = \mathcal{O}(m)$  time for all the nodes are needed [5]. ■

Each time we augment the matching, its cardinality increases by one. If the algorithm terminates, we have a maximum matching according to Lemma 3.

First we try to find an augmenting path using a labelling technique which starts at an unmatched node  $p$  and then uses a search algorithm to identify all reachable nodes. If the algorithm finds an unmatched node, it has discovered an augmenting path. If there is no such unmatched node, there is no augmenting path starting at node  $p$ .

We will grow a search tree rooted at node  $p$  such that each path in the tree from node  $p$  to another node is an alternating path. We refer to this tree as an alternating tree and nodes in the tree are labelled nodes and the others are unlabelled. The labelled nodes are of two types: even or odd. The root node is labelled with even. Notice that whenever an unmatched node has an odd label, the path joining the root node to this node is an augmenting path.

Theorem 1: Optimal Matching of Smart Dust Particles, is equivalent to the Maximal-BGMP.

Proof: Without the loss of generality we can see that the problem comprises of two sets of nodes: Senders  $S_i$  and Receivers  $R_i$  [7], [8]. The senders send out information to the receivers (in general they can be  $N$ ), these relay of information form the edges, going to the receivers. Since with the atmospheric and other constraints, all we know is the angles that the edges make with the surface where the receivers are but not the length of the edges. From Lemma 1, we are clear that if we find such a matching, the problem is equivalent to that of the BGMP. Although, we are only interested in matching a set of edges to the senders, but a more realistic question would be: How many such nodes (particles) can be matched to the receivers? (Lemma 2). Thus, it leads to the formulation of the Maximal-BGMP, and  $\mathcal{O}(n \cdot m)$  would be needed to match every sender to receiver(s) (Lemma 3). ■

Vidal et al. used only two receivers to solve a more practical problem [3]. Our generalized transformation is for  $N$  senders and  $M$  receivers, but of course the same would hold for  $N$

senders and 2 receivers.

### III. OPTIMAL NUMBER OF SMART DUST PARTICLES

We will now address the question asked by Vidal et al. For a given measurement accuracy  $\epsilon$ , what is the optimal number of leaves Smart Dust particles?

They conjectured that it would be  $n \sim \frac{1}{\epsilon}$ . Here, we will develop the necessary mathematics needed for this problem and finally formulate the formal proof to their conjecture.

Consider two different Receivers  $R_1$  and  $R_2$  on the horizontal separated by distance  $d$ . Both receive signals from the Sender  $S$  with the same wavelength  $\lambda$ . The sender  $S$  is at some vertical distance making an angle  $\theta$  with the vertical axis passing through the horizontal mid-point  $\frac{d}{2}$ . Assuming

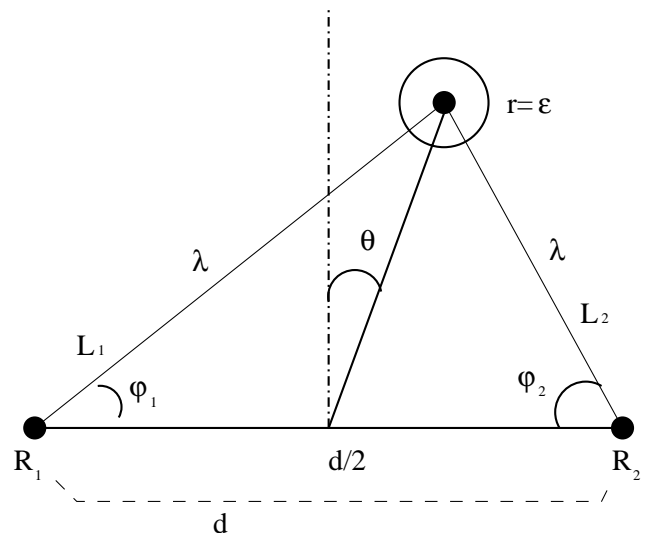


Fig. 1. Two Receivers and one Sender

that with the drift and other atmospheric constraints at any given time  $t$ , the two lengths  $L_1$  and  $L_2$  do not match. If so then we can find the difference of the arrival of the signals (phase difference). We can formulate Figure 1, as a set of complex numbers. By doing so we are able to represent the physical quantities associated with the complex numbers. The vectors become  $L_1 e^{i\phi_1}$  and  $L_2 e^{i\phi_2}$  with real part as  $L_1 \cos\phi_1$  and  $L_2 \cos\phi_2$ . By adding them together we get  $L e^{i\phi_L} = L_1 e^{i\phi_1} + L_2 e^{i\phi_2}$ . We will now find the length of  $L$ , the complex conjugate would be the same expression as that of the normal vector addition, except that the signs are reversed. We get:

$$L^2 = L_1^2 + L_2^2 + L_1 L_2 (e^{i(\phi_1 - \phi_2)} + e^{i(\phi_2 - \phi_1)}) \quad (1)$$

Now,  $e^{i\theta} + e^{-i\theta} = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta = 2\cos\theta$ . Thus we obtain:

$$L^2 = L_1^2 + L_2^2 + 2L_1 L_2 \cos(\phi_2 - \phi_1) \quad (2)$$

The resultant ensures the effects of both the receiver's capabilities  $L_1^2$  and  $L_2^2$ , along with their correction factor. This correction factor, is the interference effect. Note that this model would also result in a negative correction factor. This

can be rectified by rotating the receivers clockwise by  $\pi$ , resulting in a positive factor again. But in general since the particles, would be so densely populated, there would be enough positive factors, to cancel out the effect of the negative factors.

We will now induce errors, which are due to the drift or other unknown factors. Once such a scenario is reached, the location of the sender  $S$  is not represented by a single point, but by a sphere of radius  $\epsilon$ . Figure 2 shows such a scenario.

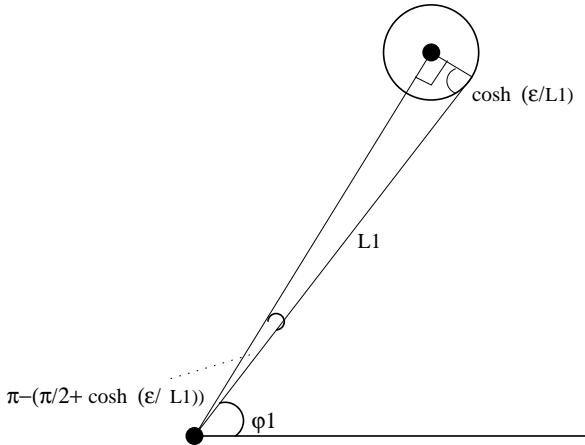


Fig. 2. Receiver and Sender with drift error  $\epsilon$

For our convenience, we draw the wedge as a right angle triangle, although this is not necessarily true in general. But since the error is of random nature the assumption of wedge being a right-angled triangle does not depict a worst case scenario. Now, we are able to compute the drift in the angle of reception for the receivers and now the vectors take the form of  $L_1 e^{i(\cosh(\frac{\epsilon}{L_1}) + \phi_1)}$  and the angle for the receiver  $R_1$ , would be anything between  $\phi_1$  to  $\pi - (\frac{\pi}{2} + \cosh(\frac{\epsilon}{L_1}))$  and similarly for receiver  $R_2$ . But due to the symmetry of the two receivers, the effect of the error in the sender's location, would be cancelled out and it can be shown that a similar result is obtained as in Equation 2.

We still have not yet been able to find the exact position of the source. But this problem is answerable by finding the phase difference  $\phi_1 - \phi_2$  (the arrival of signals at receivers  $R_1$  and  $R_2$ ). We will now derive a lemma, that is necessary for the proof that the phase difference solves the problem for finding the location of the sender  $S$ .

**Lemma 4:** If  $L_1 \neq L_2$  and receivers  $R_1$  and  $R_2$  are  $d$  distance apart, then there would be a phase delay.

**Proof:** Let there be two receivers  $R_1$  and  $R_2$  at distance  $d$  apart, receiving signals of the same amplitude from source  $S$ . Due to the distance  $d$  and the setup angle  $\theta$  over the axis at point  $\frac{d}{2}$ , one of the legs of the outer triangle would be larger than the adjacent leg connecting the other receiver. Thus, there would be an intrinsic relative phase delay  $\alpha$ , i.e. if one signal arrives at time  $t_0$ , than the other signal would arrive at time  $t_0 + \alpha$ . ■

**Theorem 2:** Finding the phase difference  $(\phi_1 - \phi_2)$  gives the generalized formula of the optimal angle required for the

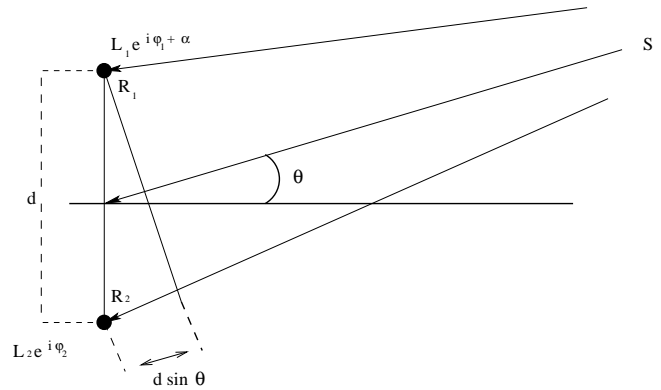


Fig. 3. Receivers with a phase difference  $\alpha$  between them

minimized interference (maximal number of senders  $S_n$ ).

**Proof:** The phase relation from Lemma 4 and Figure 3 is  $d \cdot \sin \theta$ , which is the difference in the distance from the source  $S$  to the two receivers  $R_1$  and  $R_2$ . Since the sender's location is faulty, we use the factor of  $2\pi\epsilon$  (the ball surrounding the sender in Figure 1) as error. Thus, we obtain:

$$\phi = \phi_2 - \phi_1 = 2\pi\epsilon d \sin \theta \tag{3}$$

In the case where the phase difference  $\phi = \frac{\pi}{2}$ , it follows from Equation 2, since  $\cos \frac{\pi}{2} = 0$ .

$$L^2 = L_1^2 + L_2^2 \tag{4}$$

In other words we have the exact additive distance. ■

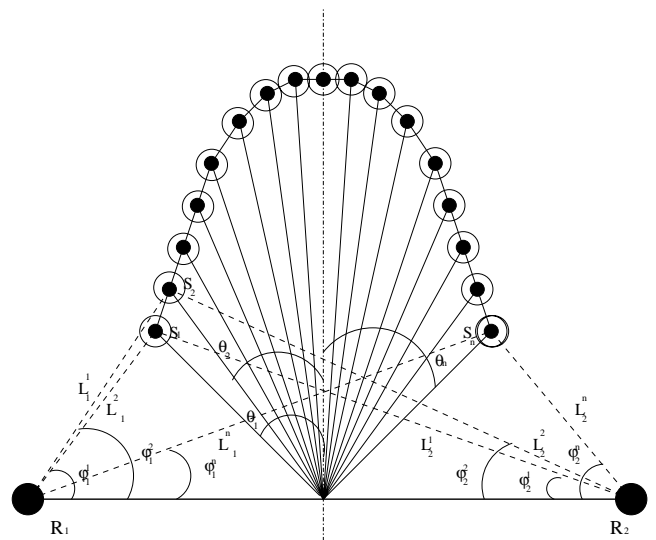


Fig. 4.  $N$  Senders and 2 Receivers

Equation 2, is none other than the well known Cosine law, and the special case Equation 4, the well known Pythagorean formula for the right-angled triangle. In fact, if  $\phi \neq \frac{\pi}{2}$ , then for the case  $\phi > \frac{\pi}{2}$ , we have  $\cos \phi < 0$ . Thus, the term  $2L_1 L_2 \cos(\phi_2 - \phi_1) < 0$  in Equation 2, i.e. a negative value (*destructive interference*) but as we had already argued that would be of little effect, as we would have a

geometric symmetry and therefore Equation 2, would look like:  $L^2 < L_1^2 + L_2^2$ . For the case  $\phi < \frac{\pi}{2}$ , since  $\cos \phi > 0$ , we will have  $2L_1L_2\cos(\phi_2 - \phi_1) > 0$  and thus it would follow from Equation 2, that in the case  $\phi < \frac{\pi}{2}$ ,  $L^2 > L_1^2 + L_2^2$ , i.e. a positive value (*constructive interference*). Thus, if we would like to have an exact measurement of the angles at which the sender should transmit data to the receiver(s), we have to set  $\phi = \frac{\pi}{2}$  in Equation 3. Thus we arrive at,

$$\sin \theta_{opt} = \frac{1}{4d\epsilon} \quad (5)$$

From Equation 5, we now derive some qualitative conclusions. First as  $\epsilon \rightarrow 0$ , notice the denominator becomes small and  $\sin \theta_{opt}$  increases. This implies that  $\theta \rightarrow \frac{\pi}{2}$ . Therefore, lower error rate are optimally better along the axis connecting the receivers. If  $\epsilon \rightarrow \infty$  the term  $\frac{1}{\epsilon}$  becomes very small and  $\sin \theta_{opt} \rightarrow 0$ . This implies that  $\theta_{opt} \rightarrow 0$ . Thus, higher values of  $\epsilon$  are better viewed optimally perpendicular to the axis connecting the receivers. It is intuitively true that the value  $4 \cdot d$  in the denominator of Equation 5, would remain constant for a given set of scenario. Therefore, it will have no effect on the outcome of the result and we can eliminate them altogether. Thus, we can say  $\sin \theta_{opt} = \frac{1}{\epsilon}$ . As there can be at most  $N$  senders spread over the horizon as shown in Figure 4, the number of senders, would be confined in a region of  $\pi$ . In Figure 4, each of the senders can be optimized by Equation 5. Optimizing the angle for each of the  $N$  senders would give the globally optimized solution for the  $N$  senders and 2 receivers problem. Thus,  $n \sim \frac{1}{\epsilon}$  holds, and Vidal et al. made the correct conjecture.

#### IV. CONCLUSION AND OPEN PROBLEMS

It might take some time before we can efficiently make use of the Smart Dust. Its application towards weather maps is much useful, but who knows maybe in the future humans would make use of these materials in some different fashion. We derived an approach for the maximal matching of these particles to the receivers, and also proved the conjecture made in [3]. Although, we do admit that the bound on this number is very loose, and we hope that in the future someone would come up with a tighter bound. Moreover, it would be interesting to see, if experimentally proven results are obtained.

#### REFERENCES

- [1] J. M. Kahn, R. H. Katz, and K. S. J. Pister, "Next century challenges: Mobile networking for smart dust," in *MOBICOM*, 1999, pp. 271–278.
- [2] L. Doherthy, "Algorithms for position and data recovery in wireless sensor networks," Master's thesis, University of California at Berkeley, June 2000.
- [3] E. Vidal, L. Longpré, V. Kreinovich, and H. Haitao, "Geoinformatics of smart dust," *Geoinformatics*, vol. 11, no. 2, pp. 54–58, 2001.
- [4] C. H. Papadimitriou and K. Steiglitz, *Combinatorial Optimization: Algorithms and Complexity*. Prentice Hall, 1982.
- [5] R. E. Tarjan, "Data structures and network algorithms," *CBMS-NSF Regional Conference Series in Applied Mathematics*, vol. 44, 1983.
- [6] N. Christofides, *Graph Theory— an Algorithmic Approach*. Academic Press, 1975.
- [7] R. K. Ahuja, T. L. Magnanti, and J. B. Orlin, *Network Flows: Theory, Algorithms and Applications*. Prentice Hall, 1993.
- [8] J. Edmonds, "Paths, trees, and flowers," *Canadian Journal of Mathematics*, vol. 17, pp. 449–467, 1965.