Integers, Game Trees and some Unknowns

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1 In the Beginning the Numbers

Over the number of years, Conway [1, 2] has presented a fruitful theory to link numbers (number theory) and combinatorial games. This theory is useful in analyzing and to some extent predicting the outcome of games. Combinatorial games require the basic assumption that all moves are visible to both of the players and chance (dice) does not play any role. Thus, a game can also be represented as a tree with left child of the tree node capturing the outcome or the effect of one player on the entire (or the subgame) game and the right child of the tree node as the play of the second player. It is to be noted that for this reason we often seen players referred to as *Left* and *Right*.

In this paper, we will introduce the link between number theory and combinatorial games, and we will also show that the current theory although very powerful, is in some cases not complete. In the next few passages, we will introduce some of the basic concepts of number theory related to combinatorial games. We will show the effectiveness of this theory by giving some examples from real board games. These examples will also lead to some open problems in Combinatorial Game Theory (CGT).

Conway [2, pp.4-6] defines (surreal) numbers as: "If L and R are any two sets of numbers, and no member of L is $\geq$ any member of R, then there is a number $\{L \mid R\}$. All numbers are constructed in this way."

From this definition a number x can be written as: $\{x^L|x^R\}$, and each of the individual elements as: x^L (left element), x^R (right element). This breakup of number(s) allow us to apply simple set operations that will mimic addition, subtraction, multiplication, etc. For example, the addition of two numbers x and y can be obtained as: $x + y = \{x^L + y, x + y^L|x^R + y, x + y^R\}$.

Here it is important to mention some of the properties that these numbers should hold:

- $x \geq y, x \leq y$:
  $x \geq y$ if and only if no $x^R \leq y$ and $x \leq$ no y^L. Similarly $x \leq y$ if and only if $y \geq x$.

- $x = y, x > y, x < y$:
  $x = y$ if and only if $x \geq y$ and $y \geq x$. $x > y$ if and only if $x \geq y$ and $y \ngeq x$. $x < y$ if and only if $y > x$.

- $-x$:
  $-x = \{-x^R| - x^L\}$.

- $x + y$:
  Already explained.

- xy:
  $xy = \{x^L y + xy^L - x^L y^L, x^R y + xy^R - x^R y^R | x^L y + xy^R - x^L y^R, x^R y + xy^L - x^R y^R\}$.

The most basic and interesting of all the numbers is the number ZERO. Based on the construction defined by Conway, every number has the form of $\{L|R\}$, where L and R are the two sets of the earlier constructed numbers (recursive definition). What we do not know at this moment is the base case for the recursive construction. But to our amazement, the basic set theory answers our concern. Before any number is constructed both the left and the right elements of the set have to be *empty*. Eureka! We have a number already. This construction $\{\ |\ \}$ is the famous number 0. We now have to see that indeed this empty construction conforms to number 0. If it does, then the rest of the numbers are just a trivial construction based on the definition.

The number 0 which evolved from the empty set construction can be justified to be indeed 0, by the following arguments. There can not exist an inequality of the form $0^L \geq 0^R$ for the simple reason that there is neither a 0^L nor 0^R. Moreover $0 \geq 0$, since there does not exist any inequality of the form of $0^R \leq 0$ or $0^L \leq 0$. By these arguments, we are hopelessly left to say that $0 = 0$, *i.e.*, the ZERO constructed by the empty set is equivalent to the 0 in number theory. Other definitions such as $-0 = 0+0 = 0$ also hold, since there can be no number of any of the forms -0^R, -0^L, $0^R + 0$, $0 + 0^R$, $0^L + 0$, and $0 + 0^L$.

We will now extend the definition of the numbers to some more integral values. Due to the shortage of space we will only consider the first two integral values, *i.e.*, 1 and -1. It is easy to not that from the recursive definition of constructing numbers, the next generation of numbers after $\{\ |\ \}$ are: $\{0|\ \}$, $\{\ |0\}$, and $\{0|0\}$. But we have already mentioned that $0 \geq 0$, therefore $\{0|0\}$ does not qualify as a number. Thus, we are left with only two cases. Out of these two cases: $\{0|\ \}$ is the number

1 and: $\{\ |0\}$ is the number -1. It is easy to follow from the arguments placed for analyzing the number 0 that $\{0|\ \}$ indeed conforms to 1. Moreover it is also clear that -1 is simply the case where x is the negative inverse of $-x$. For more details we encourage the readers to browse the text in [1].

We will now make use of real pawn games on Xiangqi board to describe some of the integral values. Before doing that we explain some of the underlying assumptions that we considered while analyzing such pawn games.

Assumption 1: The King piece is not allowed to move.

This is is actually equivalent to say that the subgame played by the two kings on their respective 3×3 roaming areas has value zero. Since the kings could move forever, resulting in a draw, it simply kills the purpose of analysis. But it is close enough since we could augment CGT to include draws by arbitrarily treating a draw as a second player win. We can support this argument by saying that at least, it is a position where the first player cannot win. We put a similar restriction on pawns. Since pawns can move horizontally after crossing the river in the middle of the board they are another potential source of loopy games.

Assumption 2: The pawns after they cross the river can only move in the horizontal direction if there is a possibility to capture an opponent's piece.

In particular, after the opponent has lost all his pieces the pawns can only move forward. This allows the pawns to move horizontally, but only in one direction until they hit the board boundary. With this assumption we will usually underestimate the value of a game position because we deprive a player of additional spare moves, but we will still be able to predict the right winner. Elkies [3] observed that another drawback of Chess is the rather small board size, compared to the Go board. In particular, most pieces can have a far reaching impact. This is also true for Xiangqi. As a consequence, it is very difficult to subdivide a game into several independent subgames, a key step in CGT analysis. Although the Xiangqi board is slightly bigger (9×10) than the Chess board (8×8) it has only five pawns besides the pairs of higher order pieces (rook, guardian, cannon, and elephant). The rules of Xiangqi [4] are more complex than in Chess and players rely more on higher order pieces rather than pawns.

Assumption 3: Subgames on different parts of the board are independent of each other.

We usually consider subgames on a single file or on a collection of files. We denote by $val(x)$ the value of the subgame on file x, and by $val(x-y)$ the subgame on files $x, x+1, \ldots, y$. The assumption that pieces in one subgame cannot interact with the pieces in another subgame is clearly a strong restriction, and it might easily result in a wrong evaluation of a game position. In Chess pawn endgames,

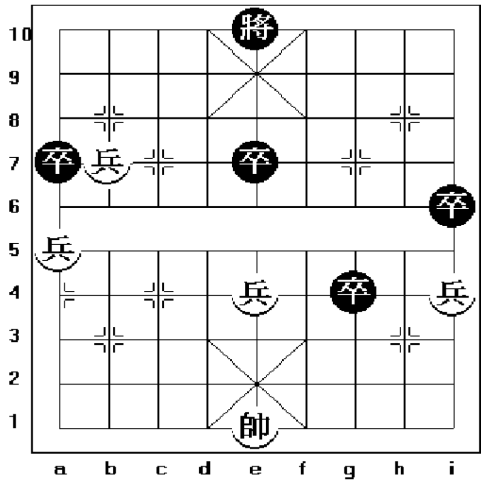

Figure 1: An integral value on board.

a subgame usually ends with the pawns blocking each other. This is not possible in Xiangqi. Therefore, we only analyze subgames until the first piece is captured.

Assumption 4: A subgame ends either when some piece is captured or when there exists no piece that can be moved.

Conceptually, after a piece has been captured all other pieces are considered immobile, so we consider this to be a subgame of value 0. We note that Elkies implicitly introduced a similar assumption when he analyzed Chess positions [3]. He lets a game end when one player could be forced to move a piece in a trébuchet (mutual Zugzwang) position, thus eventually losing the game.

2 Integers on the Xiangqi Board

In this section we will give examples of some integral values on a Xiangqi board. Compared to Chess, pawns in Xiangqi seem to be weaker because they are never elevated to higher order pieces. But their movement becomes more complicated when they cross the river, greatly increasing their power. Fortunately (or un–fortunately), pawns cannot be blocked by an opponent's pawn on the same file as capture is made on the same file. Thus, pawns will usually survive longer than in Chess.

In Figure 1, the subgame on file `i` has value $val(i) = 0$ because the first player to move immediately loses his pawn. And we can create positions with arbitrary integer values on a single file by placing one Red pawn on the black half of the

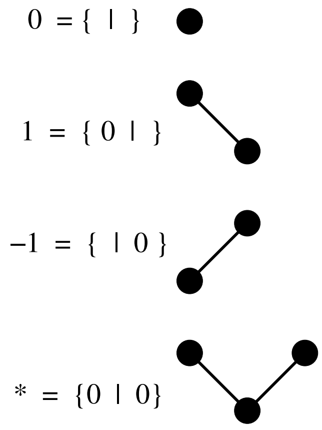

Figure 2: Simple game trees.

board and one Black pawn on the red half of the board. These pawns cannot interact, they can only move forward until they reach the baseline. The value of this subgame is then the difference between the number of Red's moves and the number of Black's moves. Having subgames of integer value on single files, we can easily create subgames of fractional value ($\frac{1}{2} = \{0|1\}$, for example). It seems impossible to create more complex positions of integral or fractional value because, by Assumption 4, a subgame ends when some piece is captured. As a consequence, there are no spare moves for the winner after capturing a piece (and the number of spare moves is intuitively the value of a winning position).

3 In the End the Game Trees

Conway extended his surreal number definition to the analysis of games. As mentioned in the introductory passage, a game can also be represented as a tree, with left and right children pertaining to the two (*Left* and *Right*) players. Conway [2, pp. 71-72] defines the games in the context of the surreal numbers as: "The games these (*Left* and *Right*) people play have *positions*, and in any position P, there are rules which restrict Left to move to any one of certain positions (typically P^L) called *Left options* of P, while Right may similarly move only to certain positions (typically P^R) called *Right options* of P. Since we are interested only in the abstract structures of games, we can regard any position P as being completely determined by its Left and Right options, and so we shall write $P = \{P^L|P^R\}$." It is necessary to introduce here (Figure 2) the game trees of some simple and trivial endgames. As was the case with numbers the first game tree we will look into is the game tree of $0 = \{\,|\,\}$. It is interesting to note that whosoever makes the first move in a 0 game, losses. This is because 0 is defined as a game where there are no legal moves (empty set of moves). The game value of $1 = \{0|\,\}$ has a legal move for Right, but there is no move for Left at any given moment of the game. Thus, Right when plays changes the current game to a game of value zero and then Left has no move at all. It is to be noted that in this case even if Left starts a game of value 1, he looses (no legal move to play). Similar arguments hold for game value -1. To make the number theory and combinatorial game link interesting we also analyze the game value of $\star$. We argued previously that $\star = \{0|0\}$ is actually not a number, but it is indeed a game (fuzzy game). The $\star$ is a win for the first player (no matter it is Left or Right). This is because whosoever plays first, changes the game value to 0 for the second player (an automatic win). In board games we often encounter these small but interesting games. Thus, it is important to capture them in the following passage:

- Game value 0 has a winning strategy for the Second player.
- Game value 1 has a winning strategy for the Right player.
- Game value -1 has a winning strategy for the Left player.
- Game value $\star$ has a winning strategy for the First player.

Conway also generalized the above mentioned basic games (G) to:

- If $G > 0$, then there is a winning strategy for the Left player.
- If $G < 0$, then there is a winning strategy for the Right player.
- If $G = 0$, then there is a winning strategy for the Second player.
- If $G||0$, then there is a winning strategy for the First player.

4 The Known and the Unknowns

Figure 3(a) (see [5, pp. 46] for a discussion of this position) is an example of a game not discussed in [1]. If Red moves first he can win by 1. `Cg3-e3`. However, if Black has the first move he can start with 1. ... `Re8-e2`. Then Red can never occupy the e file and the game ends as a draw, *i.e.*, we have a *dud*. The game graph of this game is shown in Figure 4.

We can also construct Xiangqi positions that are equivalent to positions of the game Nim. These game values play an important role in CGT and they are called nimbers [3, pp. 41]. Nim is played on heaps of pebbles. Both Left and Right play alternatingly taking away some pebbles from one of the heaps. The first player

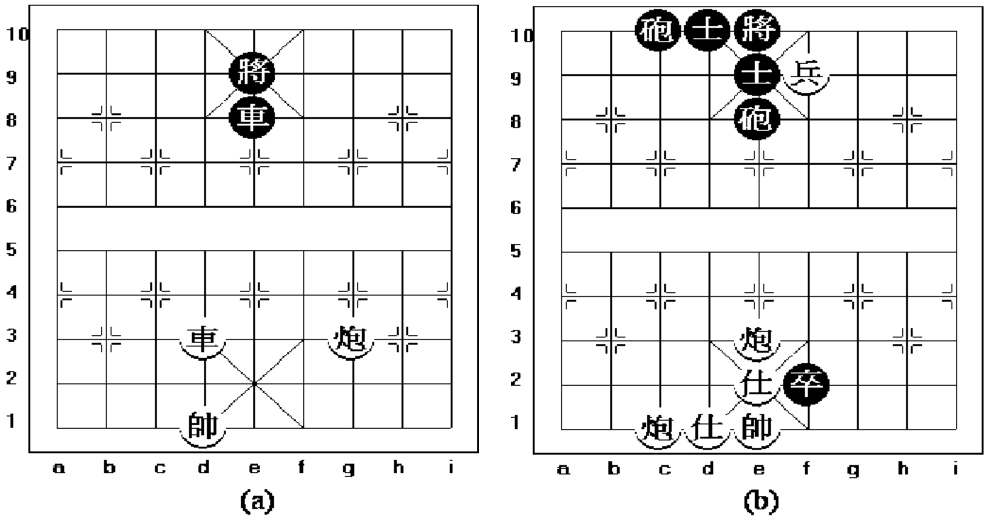

Figure 3: (a) An unknown outcome. (b) A nimber.

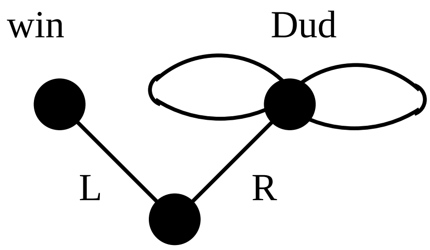

Figure 4: Game tree of the unknown outcome.

not able to move loses. In Poker–Nim [3, pp. 53], they can also put back some of the pebbles they have taken away earlier in the game. It turns out that Nim and Poker–Nim are actually equivalent. Now consider the game in Figure 3(b). Both Red and Black have two cannons and two guardians. None of the pieces are able to move (without immediately losing the game) except the cannons, and even they can only move vertically on their respective file. Thus, this position is equivalent to a Nim position. The files with the cannons correspond to heaps, and the distance between two cannons corresponds to the number of pebbles in the heap. For example, file c corresponds to a heap of size 8, and file e corresponds to a heap of size 4. It is not difficult to see that this particular position is a first player win.

5 Conclusions and Some Open Problems

In this paper we introduced the readers to the some of the basic number theory related concepts in Combinatorial Game Theory. We also showed that this theory can effectively be applied to some complex board games such as Xiangqi. However this theory is lacking some fine tuning especially in case of loopy games (board games) and some games are not defined. We ask the readers if they can answer the following posed problems.

Problem #1: Without the four assumptions described in this paper, can there be integral values found on a Xiangqi board?

Our conjecture is that without the four assumptions there is no way that a Xiangqi game can be discretely analyzed, thus it automatically falls in the category of loop games, which by definition are of infinitesimal values.

Problem #2: In Figure 4 we identified a game that is yet to be investigated in context of CGT. We ask the readers if they can formally analyze the game or in the worst case can some one name this nameless game?

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